18.03 Recitation 18, April 20, 2006

Laplace transform

Rules for the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re} s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0+)$

where we ignore singularities in derivatives at $t = 0$.

Formulas for the Laplace transform

$$\mathcal{L}[1] = 1/s \quad , \quad \mathcal{L}[e^{at}] = 1/(s-a)$$

$$\mathcal{L}[\cos(\omega t)] = s/(s^2+\omega^2) \quad , \quad \mathcal{L}[\sin(\omega t)] = \omega/(s^2+\omega^2)$$

$$\mathcal{L}[u_a(t)] = e^{-as}/s \quad , \quad \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[t^n] = n!/s^{n+1}$$

1. Use the s-shift rule and the formulas for $\mathcal{L}[\cos(\omega t)]$ and $\mathcal{L}[\sin(\omega t)]$ to find $\mathcal{L}[e^{at}\cos(\omega t)]$ and $\mathcal{L}[e^{at}\sin(\omega t)]$.

2. Find the unit impulse and unit step responses of the operator $2D + 4I$ using Laplace transform methods. What is the Laplace transform of the unit impulse and unit step response of the operator $aD + bI$ (for $a \neq 0$)?

3. Solve $2\dot{x} + 4x = e^{-t}$ with initial condition $x(0+) = 1$ using the Laplace transform.

4. Solve $2\dot{x} + 4x = e^{-t}\cos(2t)$ with $x(0+) = 0$ using the Laplace transform.